J Math Chem (2012) 50:187–209
DOI 10.1007/s10910-011-9905-5

ORIGINAL PAPER

Equivalence properties of the Cremer & Pople puckering coordinates for N-membered rings

Mathieu Kessler · José Pérez

Received: 18 July 2011 / Accepted: 24 August 2011 / Published online: 3 September 2011

Abstract The Cremer & Pople puckering coordinates are an invaluable tool in the conformational mapping and classification of rings fragments, especially for medium size rings. However, because of the topological symmetries that many chemical fragments exhibit, it is important to be aware of which values of the puckering coordinates should be considered equivalent because they correspond to the same fragments, up to a permutation of the atoms labels or a coordinates inversion. In this paper a precise description of the action of the relevant permutational operators and coordinate inversion is given for a ring of any size. The results are then specialized to the case of medium rings (6–8 atoms) for which on one hand, the symmetry equivalents of the puckering coordinates are obtained and on the other hand, in the case of structurally equivalent atoms, simple algorithms are derived to compute the representative puckering coordinates that fall into the representative asymmetric unit.

Keywords Conformational analysis · Ring fragments · Puckering coordinates

1 Introduction

The knowledge of potential-energy hyper-surface for a chemical fragment is essential for example in three-dimensional model building and related applications. To gain that

M. Kessler (✉)
Department of Applied Mathematics and Statistics, Universidad Politécnica de Cartagena,
c/ Dr Fleming, 30202 Cartagena, Spain
e-mail: mathieu.kessler@upct.es

J. Pérez
Inorganic Chemistry Group, Universidad Politécnica de Cartagena, c/ Dr Fleming, 30202 Cartagena,
Spain
e-mail: jose.pperez@upct.es

knowledge a variety of computational methods is used based on gas-phase force-field parameters, but it is also informative to analyze variations in the geometry that occur in the available crystallographic observations of the fragment. A huge collection of crystallographic data on molecular structures can be found in the Cambridge Structural Database (CSD), see Allen [1], which has been extensively searched to extract conformational information on given chemical fragments, see for example the series of papers by Allen and coauthors, [2] and [3], or [7] and [8]. Even if the associated statistical studies typically use as input data the N intra-annular torsion angles which are well understood and structurally informative, it is very common to begin the exploration of the dataset under consideration by an analysis of the Cremer & Pople coordinates, see [4], which is an alternative description of lower dimensionality, based on the out-of-plane atomic displacements (z_j) from a planar basis conformation. To support that claim, let us mention that one of the conclusions of the above mentioned series of papers on conformational mapping and classification of medium rings by Allen and coauthors is *"For ring systems, the Cremer & Pople (1975) analysis is invaluable"*, [3] p. 890.

A significant difficulty is however, met in the conformational analysis and mapping based on experimental crystallographic observations: additionally to the presence of conformational enantiomers, many of the chemically interesting fragments present topological symmetry, which implies that fragments that can be deduced from one another by atomic permutations should be considered as equivalent. There are two ways around this difficulty: either the statistical routines are modified extensively to take into account these equivalences, this is the case in [8] or in [6] for example, or the original dataset is expanded prior to a statistical analysis to include all of the conformational variants dictated by the permutational symmetry group of the fragment. The latter approach is followed for example in [3], where the dataset labelled 8C1 constituted by eight sp^3-C consisting of 32 fragments is expanded to contain 1,024 symmetry equivalents.

It is rather simple to describe the action of permutational operators or coordinate inversion on the set of torsion angles associated to a chemical fragment, see [6], it is however, more involved to obtain a similar description for their action on the Cremer & Pople coordinates. In fact, to our knowledge there is no available reference where this action is expressed for a ring of any size, or where it is clearly stated what is the precise equivalence structure of the puckering coordinates as inherited of the equivalence structure for the conformations. Some partial bits of information are scattered through several references. The purpose of this paper is to fill that gap and present on one hand a precise description of the action of the permutational operators and coordinate inversions on the Cremer & Pople puckering coordinates for a ring of any size, and on the other hand specialize these results to medium rings, where it is explained which values of puckering coordinates should be considered as equivalent, and simple algorithms are presented to derive the representative puckering coordinates. These algorithms remove in particular, as far as Cremer & Pople puckering coordinates are concerned, the need for the data expansion and allow to obtain straightforwardly what in [2], is called the asymmetric representative unit.

The structure of the paper is the following: the basic vectors, the operators and their properties are introduced in Sect. 2, the action of the permutational operators and coordinates inversion upon the Cremer & Pople coordinates is described in Sect. 3 while Sect. 4 presents the specialized results to the case of medium rings. Some concluding remarks are collected in Sect. 5. Finally, the proofs can be found in an Appendix A.

2 Preliminary notations and properties

In some cases, it is possible to choose a different starting point for the numbering of atoms in the ring. If $A_1,\ A_2,\ \ldots,\ A_N$ denote the N consecutive atoms of the ring for a chosen numbering, the ring is described as

$$A_1 A_2, \ldots, A_N.$$

If the starting point is changed from atom A_1 to atom A_2, the same ring is now described as $A_2, \ldots, A_N A_1$. Similarly, if we choose to number the atoms in the ring using the same starting point but going through the atoms using the opposite direction, the ring we describe is $A_1 A_N, \ldots, A_3 A_2$. Our purpose is to explore the consequences of these operations, together with mirroring the ring, on the computation of the associated puckering coordinates.

It turns out to be convenient, to manipulate puckering coordinates and deduce their properties, to use as a basis of $\mathbb{R}^N$ the vectors introduced in the next subsection.

2.1 The vectors

Given N, the number of atoms in the ring, the following N-dimensional vectors are introduced:

$$\text{For } m \text{ integer, } m \geq 0, \quad \begin{cases} \mathbf{e}_c(m) = \left(\cos\left(\frac{2\pi m(j-1)}{N}\right);\ j = 1, \ldots, N\right) \\ \mathbf{e}_s(m) = \left(\sin\left(\frac{2\pi m(j-1)}{N}\right);\ j = 1, \ldots, N\right) \end{cases}$$

Notice in particular that $\mathbf{e}_c(0) = (1, 1, \ldots, 1)$ and for example, if N is even, $\mathbf{e}_c(N/2) = (1, -1, 1, -1, \ldots, -1)$.

Consider the following set of vectors to construct a basis for $\mathbb{R}^N$.

- if N is odd, $\mathbf{e}_c(0)$ together with $\mathbf{e}_c(m), \mathbf{e}_s(m)$, for $m = 1, \ldots, (N-1)/2$, form a basis for $\mathbb{R}^N$.
- if N is even, $\mathbf{e}_c(0)$ together with $\mathbf{e}_c(m), \mathbf{e}_s(m)$, for $m = 1, \ldots, N/2 - 1$ and $\mathbf{e}_c(N/2)$, form a basis for $\mathbb{R}^N$.

2.2 The operators

Consider a N-dimensional vector

$$\mathbf{z} = (z_1, \ldots, z_N),$$

and introduce the following three operators that act on $\mathbf{z}$, which will be associated to the permutational operators and coordinates inversion for a chemical fragment.

– **Translation** T

$$T\mathbf{z} = (z_2, z_3, \ldots, z_N, z_1). \tag{1}$$

– **Change of direction** D

$$D\mathbf{z} = -(z_1, z_N, \ldots, z_3, z_2). \tag{2}$$

– **Mirror image** M

$$M\mathbf{z} = -(z_1, z_2, \ldots, z_{N-1}, z_N) = -\mathbf{z}. \tag{3}$$

As recalled in Sect. 3, the Cremer & Pople puckering coordinates use the vertical deviations of each atom within the ring with respect to a so-called "mean" plane which goes through the geometrical center of the ring. These vertical deviations are collected in a vector of z-coordinates, $\mathbf{z}$. In Sect. A.1.2 located in the Appendix A, it is proved that if $\mathbf{z}$ is the vector of z-coordinates of the ring described as $A_1A_2, \ldots, A_N$, $T\mathbf{z}$ defined above is the vector of z-coordinates of the ring described as $A_2A_3, \ldots, A_NA_1$, $D\mathbf{z}$ is the vector of z-coordinates of the ring described as $A_1A_N, \ldots, A_3A_2$ and finally $M\mathbf{z}$ is the vector of z-coordinates of its mirror image ring.

2.3 Action of the operators on the vectors $\mathbf{e}_c(m)$ and $\mathbf{e}_s(m)$

Lemma 1 *For all $m \in I\!N$, the following relations hold*

$$T\mathbf{e}_c(m) = \cos\left(\frac{2\pi m}{N}\right)\mathbf{e}_c(m) - \sin\left(\frac{2\pi m}{N}\right)\mathbf{e}_s(m), \tag{4}$$

$$T\mathbf{e}_s(m) = \sin\left(\frac{2\pi m}{N}\right)\mathbf{e}_c(m) + \cos\left(\frac{2\pi m}{N}\right)\mathbf{e}_s(m), \tag{5}$$

$$D\mathbf{e}_c(m) = -\mathbf{e}_c(m), \qquad D\mathbf{e}_s(m) = \mathbf{e}_s(m), \tag{6}$$

and of course,

$$M\mathbf{e}_c(m) = -\mathbf{e}_c(m), \qquad M\mathbf{e}_s(m) = -\mathbf{e}_s(m). \tag{7}$$

Proof To prove (4) and (5), it is enough to use the trigonometric formulae:

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b),$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b),$$

while (6) is deduced using

$$\cos\left(\frac{2\pi m(j-1)}{N}\right) = \cos\left(\frac{2\pi m(N-(j-1))}{N}\right),$$
$$\sin\left(\frac{2\pi m(j-1)}{N}\right) = -\sin\left(\frac{2\pi m(N-(j-1))}{N}\right).$$

3 Puckering coordinates and change of starting point, direction in the numbering of the ring and coordinates inversion

The Cremer & Pople puckering coordinates use the vertical deviations of each atom within the ring with respect to a so-called "mean" plane which goes through the geometrical center of the ring and is uniquely defined, irrespectively of the choice of the starting atom or the direction in the numbering process, and which is moreover common to the ring and its mirror image. In the Appendix A.1, the definition of the mean plane is recalled and a description in terms of the basis vectors of Sect. 2.1 is given, which allows moreover to prove the property of invariance of the mean plane with respect to changes in the numbering or when the mirror image is considered.

When considering puckering coordinates, one deliberately ignores the x and y coordinates of the atoms projections on the mean plane, and concentrates on the deviations with respect to the mean plane, i.e. the z-coordinates.

Given a N-membered ring, since the vector $\mathbf{z}$ of z-coordinates is N-dimensional, it can be described in terms of its scalar products with the vectors of any basis of $\mathbb{R}^N$. It turns out to be convenient to choose the basis described in Sect. 2.1 to describe and manipulate the puckering coordinates.

On one hand, as a consequence of the definition of the "mean" plane, with respect to which $\mathbf{z}$ contains the deviations of the atoms, it holds, see (27) in Appendix A.1,

$$\langle \mathbf{z}, \mathbf{e}_c(0)\rangle = 0, \quad \langle \mathbf{z}, \mathbf{e}_c(1)\rangle = 0, \quad \langle \mathbf{z}, \mathbf{e}_s(1)\rangle = 0,$$

where $\langle \mathbf{x}, \mathbf{y}\rangle = \sum x_i y_i$ denotes the scalar product of two vectors $\mathbf{x}$ and $\mathbf{y}$. On the other hand, the relations (12) to (14) page 1355 in Reference [4] can be written as:

If N is odd. For $m = 2, \ldots, (N-1)/2$,

$$q_m \cos\varphi_m = \sqrt{\frac{2}{N}}\langle \mathbf{z}, \mathbf{e}_c(m)\rangle,$$
$$q_m \sin\varphi_m = -\sqrt{\frac{2}{N}}\langle \mathbf{z}, \mathbf{e}_s(m)\rangle.$$

If N is even. for $m = 2, \ldots, N/2 - 1$,

$$q_m \cos\varphi_m = \sqrt{\frac{2}{N}}\langle \mathbf{z}, \mathbf{e}_c(m)\rangle,$$
$$q_m \sin\varphi_m = -\sqrt{\frac{2}{N}}\langle \mathbf{z}, \mathbf{e}_s(m)\rangle.$$

and

$$q_{N/2} = \sqrt{\frac{1}{N}}\langle \mathbf{z}, \mathbf{e}_c(N/2)\rangle.$$

It is thus deduced that indeed, if N is odd, the set of amplitude-phase pairs (q_m, φ_m) for $m = 2, \ldots, (N-1)/2$, and the set of amplitude-phase pairs (q_m, φ_m) for $m = 2, \ldots, N/2 - 1$ together with the amplitude $q_{N/2}$ if N is even, completely specifies the vertical deviations $\mathbf{z}$ with respect to the mean plane for which the relation (27) holds.

3.1 Effect of topological symmetry and of coordinate inversion on the puckering coordinates

By an abuse of notation, if q_m and φ_m denote the puckering coordinates computed from the ring $A_1A_2, \ldots, A_N$, Tq_m and $T\varphi_m$ will denote the puckering coordinates computed from the ring $A_2A_3, \ldots, A_NA_1$, Dq_m and $D\varphi_m$ will denote the puckering coordinates computed from the ring $A_1A_N, \ldots, A_3A_2$, and finally Mq_m and $M\varphi_m$ will denote the puckering coordinates computed from the mirror image ring.

Combining the previous characterization of the amplitude-phase pairs in terms of the scalar products of $\mathbf{z}$ against the vectors introduced in Sect. 2.1 with Lemma 1, the following proposition is proved to hold.

Proposition 1 *The following relations hold*

1. *For $m = 2, \ldots, (N-1)/2$ if N is odd and for $m = 2, \ldots, N/2 - 1$ if N is even,*

$$Tq_m = q_m, \qquad T\varphi_m = \varphi_m + \frac{2\pi m}{N}.$$

   Moreover, if N is even, $Tq_{N/2} = -q_{N/2}$.

2. *For $m = 2, \ldots, (N-1)/2$ if N is odd and for $m = 2, \ldots, N/2 - 1$ if N is even,*

$$Dq_m = q_m, \qquad D\varphi_m = \pi - \varphi_m.$$

   Moreover, if N is even, $Dq_{N/2} = -q_{N/2}$.

3. *Finally, for $m = 2, \ldots, (N-1)/2$ if N is odd and for $m = 2, \ldots, N/2 - 1$ if N is even,*

$$Mq_m = q_m, \qquad M\varphi_m = \pi + \varphi_m.$$

   Moreover, if N is even, $Tq_{N/2} = -q_{N/2}$.

The proof is located in the Appendix A.2.1.

4 Consequences for medium rings

In this section the general results obtained in Proposition 1 for N-membered rings are specialized to medium size rings. Concretely, a precise characterization of the equivalent pairs for puckering coordinates, together with the derivation of the asymmetric representative unit is given for rings of size six, seven or eight. Similar results can be obtained by following the same route for other ring size, however, it was decided to focus on these medium rings, for which puckering coordinates have been extensively used to extract conformational information.

4.1 Taking into account the topological symmetries

As mentioned in the introduction, because of the different possible equivalent enumerations of the atoms, topological symmetries appear for the 2D representation of the fragment which implies an equivalence structure for the puckering coordinates. In this subsection the set $\mathscr{S}$ is introduced to describe, for a given fragment, the permutations of the atoms labels which yield an equivalent 2D representation.

Consider the 2D representation of a N-membered ring fragment, denote by T and D the operators which perform the following labels permutations, respectively :

$$\begin{array}{lll} (\text{“1”}, \text{“2”}, \ldots, \text{“}N\text{”}) \xrightarrow{T} (\text{“2”}, \text{“3”}, \ldots, \text{“}N\text{”}, \text{“1”}) & \textit{Change of starting point} \\ (\text{“1”}, \text{“2”}, \ldots, \text{“}N\text{”}) \xrightarrow{D} (\text{“1”}, \text{“}N\text{”}, \text{“}N-1\text{”} \ldots, \text{“2”}) & \textit{Change of direction} \end{array}$$

Denote by $\mathscr{S}$ the set

$$\mathscr{S} = \left\{ (i, j) \text{ such that } D^j T^i \text{ leaves the 2D representation of the fragment invariant} \right\} \tag{8}$$

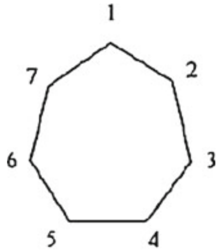
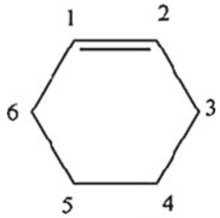
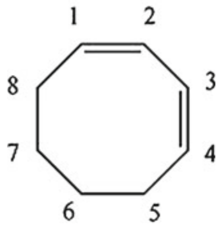
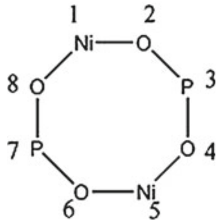
In Table 1 several examples of 2D representations of fragments and the associated $\mathscr{S}$ sets are presented. If all atoms are structurally equivalent, the $\mathscr{S}$ set associated to the N-membered ring is $\mathscr{S} = \{0, \ldots, (N-1)\} \times \{0, 1\}$.

4.2 Six-membered rings

For six-membered rings ($N = 6$), there are three puckering degrees of freedom. These are described by a single amplitude phase-pair (q_2, φ_2) and a single puckering coordinate q_3. Cremer & Pople (1975) suggest to replace these three coordinates by a “spherical polar set” (Q, θ, φ) where

$$\begin{aligned} \varphi_2 &= \varphi, \\ q_2 &= Q \sin\theta, \\ q_3 &= Q \cos\theta. \end{aligned}$$

Table 1 Examples of 2D representations of fragments and the associated sets $\mathscr{S}$ defined in (8)

	$\mathscr{S} = \{0, 1, \ldots, 6\} \times \{0, 1\}$
	$\mathscr{S} = \{(0, 0), (1, 1)\}$
	$\mathscr{S} = \{(0, 0), (3, 1)\}$
	$\mathscr{S} = \{(0, 0), (0, 1), (4, 0), (4, 1)\}$

In particular notice that $\varphi \in [0, 2\pi[$ while $\theta \in [0, \pi]$.

4.2.1 Effect of change of starting point, change of direction or mirror image

From Proposition 1, the following proposition is readily deduced

Proposition 2 *For six membered rings, the effect of change of starting point, change of direction or mirror image is the following:*

1. ***Change of starting point.***

$$T\varphi = \varphi + \frac{2\pi}{3}, \quad T\theta = \pi - \theta, \quad TQ = Q.$$

2. ***Change of direction.***

$$D\varphi = \pi - \varphi, \quad D\theta = \pi - \theta, \quad DQ = Q.$$

3. ***Mirror image.***

$$M\varphi = \pi + \varphi, \quad M\theta = \pi - \theta, \quad MQ = Q.$$

4.2.2 All possibly equivalent pairs to (φ, θ)

All the equivalent pairs for a given (φ, θ) are now to be described. Concretely, a pair (φ', θ') is said to be equivalent to (φ, θ), if there exists a composition of the operators T, D and M that allow to obtain (φ', θ') from (φ, θ). This is expressed mathematically as:
For some $r \in \mathbb{N}$, there exist for each $s = 1, \ldots, r$, a triplet $(i_s, j_s, k_s) \in \mathscr{S} \times \{0, 1\}$ such that

$$(\varphi', \theta') = \prod_{s=1}^{r} M^{k_s} D^{j_s} T^{i_s}(\varphi, \theta). \tag{9}$$

Proposition 3 *For a given pair* $(\varphi, \theta) \in [0, 2\pi[\times[0, \pi]$*, there are a total of 24 pairs candidate to be considered as equivalent. To take into account the topological symmetries, for a given* $\mathscr{S}$ *(see definition in Sect. 4.1), all pairs to be considered equivalent can be obtained from the combined action of the operators* T, D, M *resulting in* $M^k D^j T^i(\varphi, \theta)$*, for* $(i, j) \in \mathscr{S}$ *and* $k \in \{0, 1\}$*, as described in the Table 2 below.*

Table 2 Candidate equivalent pairs for (φ, θ), six-membered rings

$M^k D^j T^i(\varphi, \theta)$			
$\mathscr{S}$		$k = 0$	$k = 1$ (Mirror image)
i	j		
0	0	(φ, θ)	$(\varphi + \pi, \pi - \theta)$
1	0	$(\varphi + 2\pi/3, \pi - \theta)$	$(\varphi + 5\pi/3, \theta)$
2	0	$(\varphi + 4\pi/3, \theta)$	$(\varphi + \pi/3, \pi - \theta)$
3	0	$(\varphi, \pi - \theta)$	$(\varphi + \pi, \theta)$
4	0	$(\varphi + 2\pi/3, \theta)$	$(\varphi + 5\pi/3, \pi - \theta)$
5	0	$(\varphi + 4\pi/3, \pi - \theta)$	$(\varphi + \pi/3, \theta)$
0	1	$(\pi - \varphi, \pi - \theta)$	$(2\pi - \varphi, \theta)$
1	1	$(\pi/3 - \varphi, \theta)$	$(4\pi/3 - \varphi, \pi - \theta)$
2	1	$(5\pi/3 - \varphi, \pi - \theta)$	$(2\pi/3 - \varphi, \theta)$
3	1	$(\pi - \varphi, \theta)$	$(2\pi - \varphi, \pi - \theta)$
4	1	$(\pi/3 - \varphi, \pi - \theta)$	$(4\pi/3 - \varphi, \theta)$
5	1	$(5\pi/3 - \varphi, \theta)$	$(2\pi/3 - \varphi, \pi - \theta)$

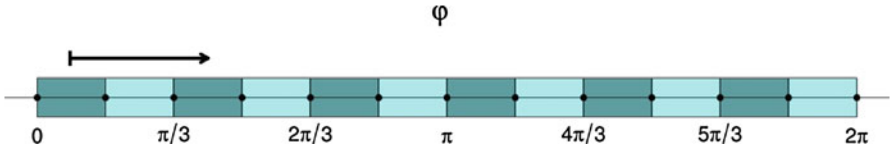


Fig. 1 The structure of equivalence for φ in $[0, 2\pi]$ for six-membered rings in the case when the constituting atoms are structurally equivalent. The *black points* correspond to symmetry centers while the *arrow* represents the translation of $\pi/3$

In the specific case when all constituting atoms are structurally equivalent within the ring, it is deduced in particular that the following values of φ *are to be considered as equivalent*

$$\begin{array}{lcccccc} & \varphi, & \varphi+\pi/3, & \varphi+2\pi/3, & \varphi+\pi, & \varphi+4\pi/3, & \varphi+5\pi/3 \\ \textit{and} & \pi/3-\varphi, & 2\pi/3-\varphi, & \pi-\varphi, & 4\pi/3-\varphi, & 5\pi/3-\varphi, & 2\pi-\varphi, \end{array}$$

where all angles are taken modulo 2π*.*

This implies an equivalence structure within the interval $[0, 2\pi]$ *for the phase angle* φ*, expressed through translations and symmetries as illustrated in Fig. 1.*

4.2.3 Obtaining the representative puckering coordinates for six-membered rings with structurally equivalent atoms

Using the results of the previous subsection, it is possible to deduce a simple algorithm to places the puckering coordinates of any six-membered chemical fragment into a arbitrarily chosen representative asymmetric unit. This point is illustrated by considering the simpler case when all constituting atoms are structurally equivalent.

Corollary 1 *The angle* φ *can be chosen to vary in* $[0, \pi/6]$ *and* θ *to vary in* $[0, \pi/2]$*. For any pair* $(\varphi, \theta) \in [0, 2\pi[\times[0, \pi]$*, its representative element can be chosen to lie in* $[0, \pi/6] \times [0, \pi/2]$*. Concretely, given* $(\varphi, \theta) \in [0, 2\pi[\times[0, \pi]$*, its representative element* $(\bar{\varphi}, \bar{\theta})$ *is obtained through:*

– *Let* $\varphi' = \varphi \mod \pi/3$*,*

$$\bar{\varphi} = \begin{cases} \varphi' & \text{if} \quad \varphi' \leq \pi/6 \\ \pi/3 - \varphi' & \text{if} \quad \varphi' > \pi/6 \end{cases}$$

– *As for* $\bar{\theta}$*,*

$$\bar{\theta} = \begin{cases} \theta & \text{if} \quad \theta \leq \pi/2 \\ \pi - \theta & \text{if} \quad \theta > \pi/2 \end{cases}$$

As a consequence, the representative unit can be taken as a $1/12$th of the half polar sphere.

4.3 Seven-membered rings

For seven-membered rings ($N = 7$), there are four puckering degrees of freedom. These are described by two amplitude-phase pairs (q_2, φ_2) and (q_3, φ_3).

4.3.1 Effect of change of starting point, change of direction or mirror image

From Proposition 1, the following proposition is deduced

Proposition 4 *For seven-membered rings, the effect of change of starting point, change of direction or mirror image is the following:*

1. ***Change of starting point.***

$$T\varphi_2 = \varphi_2 + \frac{4\pi}{7}, \quad T\varphi_3 = \varphi_3 + \frac{6\pi}{7}, \tag{10}$$

$$Tq_2 = q_2, \quad Tq_3 = q_3. \tag{11}$$

2. ***Change of direction.***

$$D\varphi_2 = \pi - \varphi_2, \quad D\varphi_3 = \pi - \varphi_3, \tag{12}$$

$$Dq_2 = q_2, \quad Dq_3 = q_3. \tag{13}$$

3. ***Mirror image.***

$$M\varphi_2 = \varphi_2 + \pi, \quad M\varphi_3 = \varphi_3 + \pi, \tag{14}$$

$$Mq_2 = q_2, \quad Mq_3 = q_3 \tag{15}$$

4.3.2 All equivalent pairs to (φ_2, φ_3)

As in the previous sections, a pair (φ_2', φ_3') is considered to be equivalent to (φ_2, φ_3), if there exists a composition of the operators T, D and M that allow to obtain (φ_2', φ_3') from (φ_2, φ_3). This is expressed mathematically as:
For some $r \in \mathbb{N}$, there exist for each $s = 1, \ldots, r$, a triplet $(i_s, j_s, k_s) \in \mathscr{S} \times \{0, 1\}$ such that

Table 3 Candidate equivalent pairs for (φ_2, φ_3) for seven membered rings

$\mathscr{S}$		$M^k D^j T^i(\varphi_2, \varphi_3)$	
i	j	$k = 0$	$k = 1$ (Mirror image)
0	0	(φ_2, φ_3)	$(\pi + \varphi_2, \pi + \varphi_3)$
1	0	$(4\pi/7 + \varphi_2, 6\pi/7 + \varphi_3)$	$(11\pi/7 + \varphi_2, 13\pi/7 + \varphi_3)$
2	0	$(8\pi/7 + \varphi_2, 12\pi/7 + \varphi_3)$	$(\pi/7 + \varphi_2, 5\pi/7 + \varphi_3)$
3	0	$(12\pi/7 + \varphi_2, 4\pi/7 + \varphi_3)$	$(5\pi/7 + \varphi_2, 11\pi/7 + \varphi_3)$
4	0	$(2\pi/7 + \varphi_2, 10\pi/7 + \varphi_3)$	$(9\pi/7 + \varphi_2, 3\pi/7 + \varphi_3)$
5	0	$(6\pi/7 + \varphi_2, 2\pi/7 + \varphi_3)$	$(13\pi/7 + \varphi_2, 9\pi/7 + \varphi_3)$
6	0	$(10\pi/7 + \varphi_2, 8\pi/7 + \varphi_3)$	$(3\pi/7 + \varphi_2, \pi/7 + \varphi_3)$
0	1	$(\pi - \varphi_2, \pi - \varphi_3)$	$(2\pi - \varphi_2, 2\pi - \varphi_3)$
1	1	$(3\pi/7 - \varphi_2, \pi/7 - \varphi_3)$	$(10\pi/7 - \varphi_2, 8\pi/7 - \varphi_3)$
2	1	$(13\pi/7 - \varphi_2, 9\pi/7 - \varphi_3)$	$(6\pi/7 - \varphi_2, 2\pi/7 - \varphi_3)$
3	1	$(9\pi/7 - \varphi_2, 3\pi/7 - \varphi_3)$	$(2\pi/7 - \varphi_2, 10\pi/7 - \varphi_3)$
4	1	$(5\pi/7 - \varphi_2, 11\pi/7 - \varphi_3)$	$(12\pi/7 - \varphi_2, 4\pi/7 - \varphi_3)$
5	1	$(\pi/7 - \varphi_2, 5\pi/7 - \varphi_3)$	$(8\pi/7 - \varphi_2, 12\pi/7 - \varphi_3)$
6	1	$(11\pi/7 - \varphi_2, 13\pi/7 - \varphi_3)$	$(4\pi/7 - \varphi_2, 6\pi/7 - \varphi_3)$

$$\left(\varphi_2', \varphi_3'\right) = \prod_{s=1}^{r} M^{k_s} D^{j_s} T^{i_s}(\varphi_2, \varphi_3). \tag{16}$$

Proposition 5 *For a given pair $(\varphi_2, \varphi_3) \in [0, 2\pi[^2$, there are a total of 28 candidate pairs to be considered as equivalent. To take into account the topological symmetries, for a given $\mathscr{S}$ (see definition in Sect. 4.1), all pairs to be considered equivalent can be obtained from the combined action of the operators T, D, M resulting in $M^k D^j T^i(\varphi_2, \varphi_3)$, for $(i, j) \in \mathscr{S}$ and $k \in \{0, 1\}$, as described in the Table 3.*

As for six-membered rings, in the case of structurally equivalent constituting atoms, an equivalence structure in $[0, 2\pi] \times [0, 2\pi]$ for (φ_2, φ_3) is deduced which is expressed through translations and symmetries, as represented in Fig. 2.

4.3.3 Obtaining the representative puckering coordinates for seven-membered rings with structurally equivalent atoms

Because of the involved equivalence structure present in $(\varphi_2, \varphi_3) \in [0, 2\pi]^2$ and illustrated visually in Fig. 2, a natural question arises: which is the smaller representative asymmetric unit for (φ_2, φ_3)? The answer can be found in the next corollary.

Corollary 2 *The parallelogram*

$$0 \le 5\varphi_2 - \varphi_3 \le \pi \quad \text{and} \quad 0 \le -\varphi_2 + 3\varphi_3 \le \pi,$$

which is represented in Fig. 2 can be chosen as representative unit.

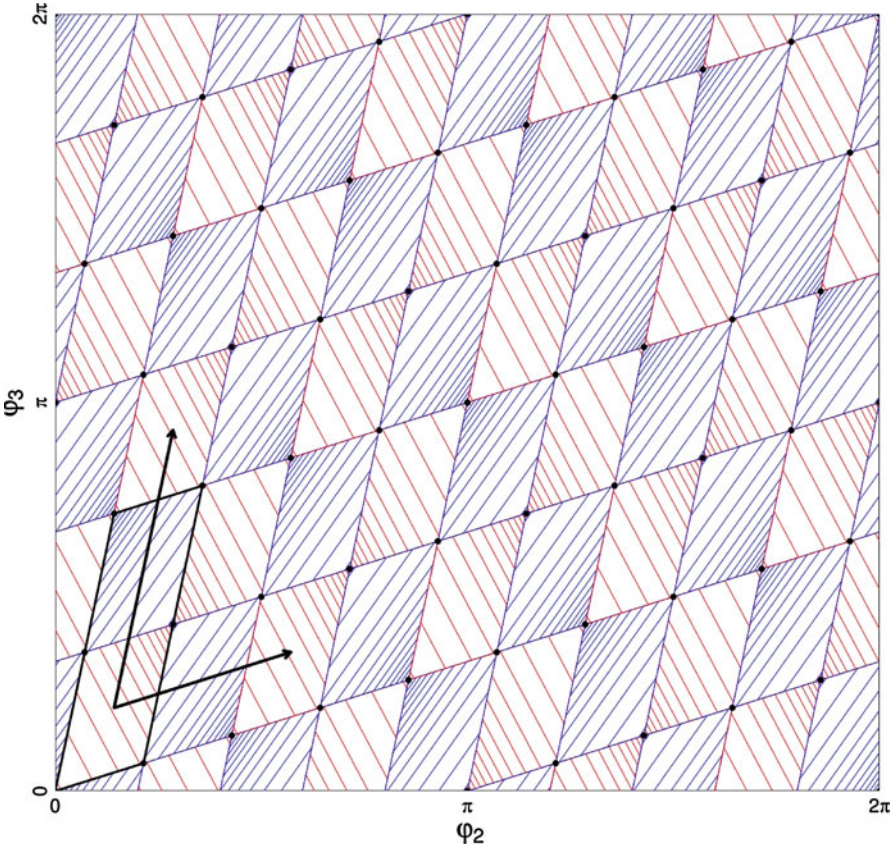


Fig. 2 The structure of equivalence for (φ_2, φ_3) in $[0, 2\pi]^2$ for seven-membered rings in the case when the constituting atoms are structurally equivalent. The representative unit is taken to be the *black-border parallelogram*. The *black points* correspond to symmetry centers, while the two *arrows* represent translations. The density of the filling lines in the parallelograms is not uniform to make it easier to appreciate visually the existing symmetries

Concretely, given $(\varphi_2, \varphi_3) \in [0, 2\pi[^2$, *its equivalent representative element* $(\bar{\varphi}_2, \bar{\varphi}_3)$ *is obtained through the following algorithm:*

1. *Let* $\psi_2 = 5\varphi_2 - \varphi_3$ *and* $\psi_3 = -\varphi_2 + 3\varphi_3$.
2. *Let* $\psi_2' = \psi_2 \mod (2\pi)$ *and* $\psi_3' = \psi_3 \mod (\pi)$

$$\left(\psi_2'', \psi_3''\right) = \begin{cases} \left(\psi_2', \psi_3'\right) & \textit{if} \quad \psi_2' \le \pi \\ \left(2\pi - \psi_2', 2\pi - \psi_3'\right) & \textit{if} \quad \psi_2' > \pi. \end{cases}$$

3. *Finally*

$$\bar{\varphi}_2 = \frac{3}{14}\psi_2'' + \frac{1}{14}\psi_3'', \quad \bar{\varphi}_3 = \frac{1}{14}\psi_2'' + \frac{5}{14}\psi_3''$$

4.4 Eight-membered rings

For eight-membered rings ($N = 8$), there are five puckering degrees of freedom. These are described by two amplitude-phase pairs (q_2, φ_2) and (q_3, φ_3) together with a single puckering coordinate q_4. In Reference [5], it is suggested to introduce the quantities (Q, θ) where

$$Q = \sqrt{q_2^2 + q_3^2 + q_4^4},$$
$$\theta \in (0, \pi), \quad q_4 = Q \cos\theta,$$

4.4.1 Effect of change of starting point, change of direction or mirror image

From Proposition 1, the following proposition is deduced

Proposition 6 *For eight membered rings, the effect of change of starting point, change of direction or mirror image is the following:*

1. ***Change of starting point.***

$$T\varphi_2 = \varphi_2 + \frac{\pi}{2}, \quad T\varphi_3 = \varphi_3 + \frac{3\pi}{4}, \quad T\theta = \pi - \theta, \tag{17}$$
$$Tq_2 = q_2, \quad Tq_3 = q_3. \tag{18}$$

2. ***Change of direction.***

$$D\varphi_2 = \pi - \varphi_2, \quad D\varphi_3 = \pi - \varphi_3, \quad D\theta = \pi - \theta, \tag{19}$$
$$Dq_2 = q_2, \quad Dq_3 = q_3. \tag{20}$$

3. ***Mirror image.***

$$M\varphi_2 = \varphi_2 + \pi, \quad M\varphi_3 = \varphi_3 + \pi, \quad M\theta = \pi - \theta, \tag{21}$$
$$Mq_2 = q_2, \quad Mq_3 = q_3. \tag{22}$$

4.4.2 All equivalent triplets to $(\varphi_2, \varphi_3, \theta)$

In this subsection, all the equivalent triplets for a given $(\varphi_2, \varphi_3, \theta)$ are identified. Concretely, a triplet $\left(\varphi_2', \varphi_3', \theta'\right)$ is said to be equivalent to $(\varphi_2, \varphi_3, \theta)$, if there exists a composition of the operators T, D and M that allow to obtain $\left(\varphi_2', \varphi_3', \theta'\right)$ from $(\varphi_2, \varphi_3, \theta)$. This is expressed mathematically as:
For some $r \in \mathbb{N}$, there exist for each $s = 1, \ldots, r$, a triplet $(i_s, j_s, k_s) \in \mathscr{S} \times 0, 1$ such that

$$\left(\varphi_2', \varphi_3', \theta'\right) = \prod_{s=1}^{r} M^{k_s} D^{j_s} T^{i_s} (\varphi_2, \varphi_3, \theta). \tag{23}$$

Table 4 Candidate equivalent triplets for $(\varphi_2, \varphi_3, \theta)$ for eight membered rings

$M^k D^j T^i(\varphi_2, \varphi_3, \theta)$			
$\mathscr{S}$		$k = 0$	$k = 1$ (Mirror image)
i	j		
0	0	$(\varphi_2, \varphi_3, \theta)$	$(\pi + \varphi_2, \pi + \varphi_3, \pi - \theta)$
1	0	$(\pi/2 + \varphi_2, 3\pi/4 + \varphi_3, \pi - \theta)$	$(3\pi/2 + \varphi_2, 7\pi/4 + \varphi_3, \theta)$
2	0	$(\pi + \varphi_2, 3\pi/2 + \varphi_3, \theta)$	$(\varphi_2, \pi/2 + \varphi_3, \pi - \theta)$
3	0	$(3\pi/2 + \varphi_2, \pi/4 + \varphi_3, \pi - \theta)$	$(\pi/2 + \varphi_2, 5\pi/4 + \varphi_3, \theta)$
4	0	$(\varphi_2, \pi + \varphi_3, \theta)$	$(\pi + \varphi_2, \varphi_3, \pi - \theta)$
5	0	$(\pi/2 + \varphi_2, 7\pi/4 + \varphi_3, \pi - \theta)$	$(3\pi/2 + \varphi_2, 3\pi/4 + \varphi_3, \theta)$
6	0	$(\pi + \varphi_2, \pi/2 + \varphi_3, \theta)$	$(\varphi_2, 3\pi/2 + \varphi_3, \pi - \theta)$
7	0	$(3\pi/2 + \varphi_2, 5\pi/4 + \varphi_3, \pi - \theta)$	$(\pi/2 + \varphi_2, \pi/4 + \varphi_3, \theta)$
0	1	$(\pi - \varphi_2, \pi - \varphi_3, \pi - \theta)$	$(2\pi - \varphi_2, 2\pi - \varphi_3, \theta)$
1	1	$(\pi/2 - \varphi_2, \pi/4 - \varphi_3, \theta)$	$(3\pi/2 - \varphi_2, 5\pi/4 - \varphi_3, \pi - \theta)$
2	1	$(2\pi - \varphi_2, 3\pi/2 - \varphi_3, \pi - \theta)$	$(\pi - \varphi_2, \pi/2 - \varphi_3, \theta)$
3	1	$(3\pi/2 - \varphi_2, 3\pi/4 - \varphi_3, \theta)$	$(\pi/2 - \varphi_2, 7\pi/4 - \varphi_3, \pi - \theta)$
4	1	$(\pi - \varphi_2, 2\pi - \varphi_3, \pi - \theta)$	$(2\pi - \varphi_2, \pi - \varphi_3, \theta)$
5	1	$(\pi/2 - \varphi_2, 5\pi/4 - \varphi_3, \theta)$	$(3\pi/2 - \varphi_2, \pi/4 - \varphi_3, \pi - \theta)$
6	1	$(2\pi - \varphi_2, \pi/2 - \varphi_3, \pi - \theta)$	$(\pi - \varphi_2, 3\pi/2 - \varphi_3, \theta)$
7	1	$(3\pi/2 - \varphi_2, 7\pi/4 - \varphi_3, \theta)$	$(\pi/2 - \varphi_2, 3\pi/4 - \varphi_3, \pi - \theta)$

Proposition 7 *For a given triplet $(\varphi_2, \varphi_3, \theta) \in [0, 2\pi[^2 \times [0, \pi]$, there are a total of 32 triplets that are candidate to be considered as equivalent (modulo 2π), To take into account the topological symmetries, for a given $\mathscr{S}$ (see definition in Sect. 4.1), all triplets to be considered equivalent can be obtained from the combined action of the operators T, D, M resulting in $M^k D^j T^i(\varphi_2\varphi_3, \theta)$, for $(i, j) \in \mathscr{S}$ and $k \in \{0, 1\}$, as described in the Table 4 above.*

In the case when the constituting atoms are structurally equivalent, an equivalence structure for (φ_2, φ_3) is deduced within $[0, 2\pi] \times [0, 2\pi]$, which can be expressed through translations and symmetries, as represented in Fig. 3.

4.4.3 Obtaining the representative puckering coordinates for eight-membered rings

Corollary 3 *The pair (φ_2, φ_3) can be restricted to vary in $[0, \pi/4] \times [0, \pi/2]$. For any triplet $(\varphi_2, \varphi_3, \theta) \in [0, 2\pi[\times[0, 2\pi[\times[0, \pi]$, its representative element is taken to lie within $[0, \pi/4] \times [0, \pi/2] \times [0, \pi]$. Concretely, given $(\varphi_2, \varphi_3, \theta) \in [0, 2\pi[\times[0, \pi]$, its representative element $(\bar{\varphi}_2, \bar{\varphi}_3, \bar{\theta})$ is obtained through:*

1. *Let $\varphi_2' = (\varphi_2 \mod \pi)$ and $\varphi_3' = (\varphi_3 \mod \pi/2)$.*
   Set moreover $m_2' = (\varphi_2\, div\, \pi)$ and $m_3' = (\varphi_3\, div\, \pi/2)$, and $\theta' = D^{m_2' + m_3'}\theta$, where div denotes the integer division.

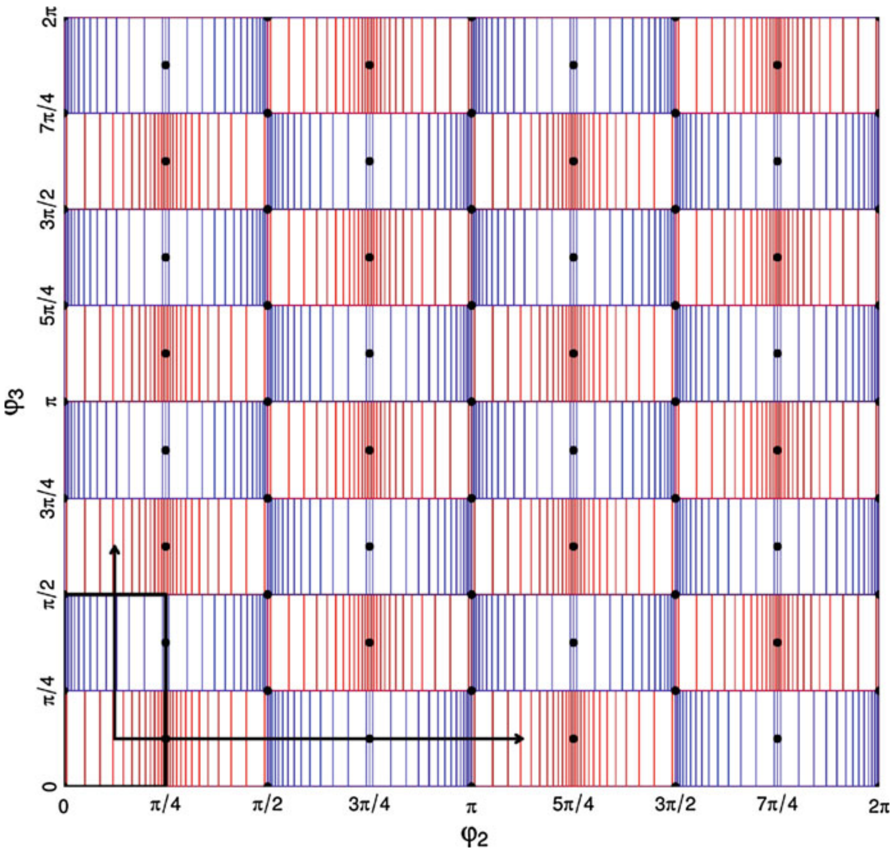


Fig. 3 The structure of equivalence for (φ_2, φ_3) in $[0, 2\pi]^2$ for eight-membered rings in the case when the constituting atoms are structurally equivalent. The representative unit is taken to be the *black-border rectangle* $(\varphi_2, \varphi_3) \in [0, \pi/4] \times [0, \pi/2]$. The *black points* correspond to symmetry centers, while the two *arrows* represent translations. The density of the filling lines in the rectangles is not uniform to make it easier to appreciate visually the existing symmetries

2. *Let*

$$\left(\varphi_2'', \varphi_3''\right) = \begin{cases} \left(\varphi_2', \varphi_3'\right) & if \;\; \varphi_2' \leq \pi/2 \\ \left(\pi - \varphi_2', \pi/2 - \varphi_3'\right) & if \;\; \varphi_2' > \pi/2. \end{cases}$$

3. *Finally*

$$(\bar{\varphi}_2, \bar{\varphi}_3) = \begin{cases} \left(\varphi_2'', \varphi_3''\right) & if \;\; \varphi_2'' \leq \pi/4 \\ \left(\pi/2 - \varphi_2'', \pi/4 - \varphi_3''\right) & if \;\; \varphi_2'' > \pi/4 \;\; and \;\; \varphi_3'' \leq \pi/4. \\ \left(\pi/2 - \varphi\prime_2, 3\pi/4 - \varphi_3''\right) & if \varphi_2'' > \pi/4 \;\; and \;\; \varphi_3'' > \pi/4 \end{cases}$$

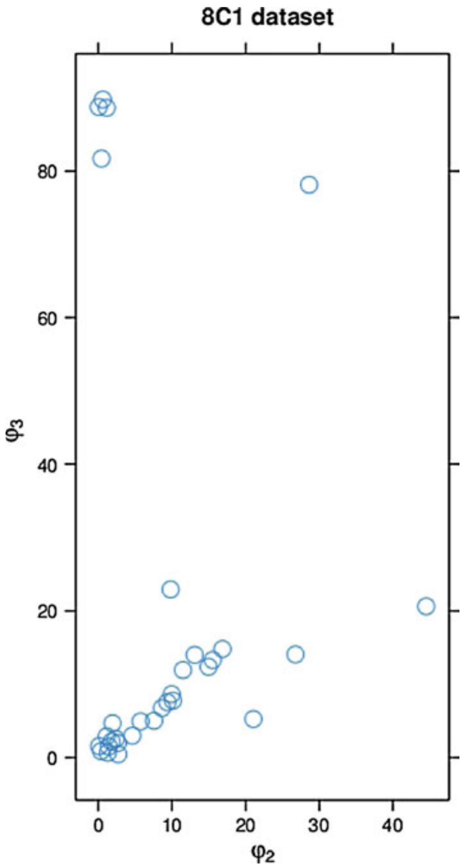


Fig. 4 φ_2, φ_3 plot for the 8C1 dataset which is analyzed by expansion in [3]. It has to be compared with plot b), Figure 5 of Reference [3]

In the case when $\varphi_2'' > \pi/4$ *and* $\varphi_3'' > \pi/4$, *set moreover* $\bar{\theta} = \pi - \theta'$ *else set* $\bar{\theta} = \theta'$.

As an illustration, the representative elements of (φ_2, φ_3) for the fragments included in the 8C1 dataset in [3] were computed following the algorithm described in the Corollary 3. In that paper, the authors carry out an expansion of the initial dataset, by considering all possible permutations, change of directions and mirror images of the fragments. From the initial 32 data-points, they build therefore, for its analysis an expanded dataset which consists of 1,024 symmetry equivalents. In particular the plot b) in Figure 5 in Reference [3] represent φ_3 versus φ_2 for the expanded 8C1 dataset, with values of φ_2 and φ_2 ranging from 0 to 360°. Using Corollary 3, as for puckering coordinates are concerned, there is no need for expansion since it explains how to obtain the representative puckering coordinates. In Fig. 4, the representative puckering coordinates are plotted for the initial 8C1 dataset, the range of values for φ_2 and φ_3 being $[0, \pi/4]$ and $[0, \pi/2]$, respectively. It is checked that it indeed corresponds to the rectangle $[0, 45°] \times [0, 90°]$ in the plot b), Figure 5 of Reference [3].

5 Conclusions

In this paper a precise description of the influence of permutations of the atoms labels or coordinates inversion onto the Cremer & Pople puckering coordinates is given for a ring of any size. The results are specialized to medium size rings for which it is deduced which values of the puckering coordinates should be considered as equivalent, after taking into account the topological symmetries present in the 2D representation of the considered fragments.

In the case when all constituting atoms are structurally equivalent, it is then possible to identify a representative unit and provide a simple way to compute the representative puckering coordinates of a fragment. This procedure allows to avoid the expansion of the original dataset prior to a statistical analysis as required by several conformational studies published in the literature, to include all of the conformational variants dictated by the permutational symmetry group of the fragment. Statistical procedures or graphs can then be applied on the representative elements in order to extract useful conformational information. In the case when the constituting atoms are not structurally equivalent, similar procedures can be deduced following the same lines, but they have to be worked out on a case-by-case basis.

Finally it should be mentioned that, following the same steps, results of the same nature can be obtained for higher size rings, although the puckering coordinates are much less intuitive and useful for conformational purposes in these settings.

A Appendix

A.1 The mean plane

A.1.1 Definition

For a N-membered ring, start from the atomic coordinates $(X_i, Y_i, Z_i),\ i = 1, \ldots, N$. The mean plane as introduced by Cremer & Pople in Reference [4], goes through the geometrical center of the atoms $(\bar{X}, \bar{Y}, \bar{Z})$ and admits as a normal vector the vector $\mathbf{n}$ which is defined as

$$\mathbf{n} = \frac{R' \times R''}{|R' \times R''|},$$

where

$$R' = \sum_{i=1}^{N} R_i \sin\left(\frac{2\pi(i-1)}{N}\right)$$
$$R'' = \sum_{i=1}^{N} R_i \cos\left(\frac{2\pi(i-1)}{N}\right),$$

and R_i denotes the vector position of atom i in the ring, with respect to the geometrical center, i.e $R_i = (X_i, Y_i, Z_i) - (\bar{X}, \bar{Y}, \bar{Z})$. The orientation of $\mathbf{n}$ defines a topside of the ring (above the face with clockwise numbering). The z coordinates are the vertical deviations of the atoms with respect to the mean plane, which implies that

$$\sum_{i=1}^{N} z_i = 0, \tag{24}$$

$$\sum_{i=1}^{N} z_i \cos\left(\frac{2\pi(i-1)}{N}\right) = 0, \tag{25}$$

$$\sum_{i=1}^{N} z_i \sin\left(\frac{2\pi(i-1)}{N}\right) = 0. \tag{26}$$

Reversely, these three equations define uniquely the mean plane and can be written compactly in terms of the vectors defined in Sect. 2.1:

$$\langle \mathbf{z}, \mathbf{e}_c(0)\rangle = 0, \quad \langle \mathbf{z}, \mathbf{e}_c(1)\rangle = 0, \quad \langle \mathbf{z}, \mathbf{e}_s(1)\rangle = 0. \tag{27}$$

A.1.2 Property

The mean plane only depends on the conformation of the ring and does not depend on the choice of the first atom in the numbering of the molecule, the direction that is used in the numbering; moreover a ring and its mirror image share the same mean plane, as will be proved.

Change the starting point In some cases, it is possible to choose a different starting point for the numbering of atoms in the ring. If $A_1, A_2, \ldots, A_N$ denote the N consecutive atoms of the ring for a chosen numbering, the ring is described as

$$A_1 A_2, \ldots, A_N.$$

If the starting point is changed from atom A_1 to atom A_2, the same ring is now described as

$$A_2, \ldots, A_N A_1.$$

Lemma 2 *The vector* $\mathbf{n}$ *computed using the numbering* $A_1 A_2, \ldots, A_N$ *is orthogonal to the mean plane computed using the numbering* $A_2, \ldots, A_N A_1$.

Proof If $\mathbf{z}$ is the vector of z-coordinates computed using the numbering $A_1 A_2, \ldots, A_N$, $T\mathbf{z} = (z_2, z_3, \ldots, z_N, z_1)$ is the vector of coordinates of the ring $A_2, \ldots, A_N A_1$ with respect to the same origin and vector $\mathbf{n}$. Now relations (24–26) imply that

$$\langle T\mathbf{z}, T\mathbf{e}_c(0)\rangle = 0, \quad \langle T\mathbf{z}, T\mathbf{e}_c(1)\rangle = 0, \quad \langle T\mathbf{z}, T\mathbf{e}_s(1)\rangle = 0.$$

It is enough to use (4) together with (5) to deduce

$$\langle T\mathbf{z}, \mathbf{e}_c(0)\rangle = 0, \quad \langle T\mathbf{z}, \mathbf{e}_c(1)\rangle = 0, \quad \langle T\mathbf{z}, \mathbf{e}_s(1)\rangle = 0.$$

This ends the proof.

Since Eqs. (24–26) define the mean plane uniquely, it is deduced in particular that the mean plane is the same irrespective of the choice of the first atom in the numbering of the atoms in the ring.

Change the direction If we choose to number the atoms in the ring using the same starting point but going through the atoms using the opposite direction, the ring we describe is $A_1A_N, \ldots, A_3A_2$.

Lemma 3 *The vector* **n** *computed using the numbering* $A_1A_2, \ldots, A_N$ *is the opposite of the vector* **n** *computed using the numbering* $A_2, \ldots, A_NA_1$.

Proof If **z** is the vector of z-coordinates computed using the numbering $A_1A_2, \ldots, A_N$, $D\mathbf{z} = -(z_1, z_N, \ldots, z_3, z_2)$ is the vector of coordinates of the ring $A_1A_N, \ldots, A_3A_2$ with respect to the same origin and vector -**n**. Now relations (24–26) imply that

$$\langle D\mathbf{z}, D\mathbf{e}_c(0)\rangle = 0, \quad \langle D\mathbf{z}, D\mathbf{e}_c(1)\rangle = 0, \quad \langle D\mathbf{z}, D\mathbf{e}_s(1)\rangle = 0.$$

It is enough to use (6) to deduce

$$\langle D\mathbf{z}, \mathbf{e}_c(0)\rangle = 0, \quad \langle D\mathbf{z}, \mathbf{e}_c(1)\rangle = 0, \quad \langle D\mathbf{z}, \mathbf{e}_s(1)\rangle = 0.$$

This ends the proof.

As a consequence, the effect of changing the direction of numbering in the ring does not change the mean plane but changes the sign of the vector n (and consequently of the z-coordinates).

Considering the mirror image Consider a mirror image of the ring $A_1A_2, \ldots, A_N$. A mirror image of the ring is obtained as the symmetric image of the 3D ring with respect to any plane. Two mirror images of the same ring can be made to match using a rotation and a translation. The mean planes corresponding to two different mirror images can therefore, match after the rotation and translation, which implies that the z-coordinates for two different mirror-images of the same ring will be the same.

Consider the mirror image obtained when the symmetry plane is chosen to be precisely the mean plane, it is then clear that the **n** vector of the mirror image ring is opposite to that of the original ring, which implies that the z-coordinates are opposite also.

Summary

1. **Change of starting point.** If **z** is the vector of z-coordinates of the ring described as $A_1A_2, \ldots, A_N$, $T\mathbf{z}$ is the vector of z-coordinates of the ring described as $A_2A_3, \ldots, A_NA_1$.

2. **Change of direction.** If $\mathbf{z}$ is the vector of z-coordinates of the ring described as $A_1A_2, \ldots, A_N$, $D\mathbf{z}$ is the vector of z-coordinates of the ring described as $A_1A_N, \ldots, A_3A_2$.
3. **Mirror image** If $\mathbf{z}$ is the vector of z-coordinates of the ring described as $A_1A_2, \ldots, A_N$, $M\mathbf{z}$ is the vector of z-coordinates of its mirror image ring.

A.2 Proofs

A.2.1 Proof of Proposition 1

Proof **Proof of point 1**. We have

$$q_m \cos\varphi_m = \sqrt{\frac{2}{N}}\langle \mathbf{z}, \mathbf{e}_c(m)\rangle = \sqrt{\frac{2}{N}}\langle T\mathbf{z}, T\mathbf{e}_c(m)\rangle,$$
$$q_m \sin\varphi_m = -\sqrt{\frac{2}{N}}\langle \mathbf{z}, \mathbf{e}_s(m)\rangle = -\sqrt{\frac{2}{N}}\langle T\mathbf{z}, T\mathbf{e}_s(m)\rangle.$$

From this system, using relations (4) and (5), the following equations are deduced:

$$q_m \cos\varphi_m = \cos\left(\frac{2\pi m}{N}\right)\sqrt{\frac{2}{N}}\langle T\mathbf{z}, \mathbf{e}_c(m)\rangle + \sin\left(\frac{2\pi m}{N}\right)\left(-\sqrt{\frac{2}{N}}\langle T\mathbf{z}, \mathbf{e}_s(m)\rangle\right),$$
$$q_m \sin\varphi_m = -\sin\left(\frac{2\pi m}{N}\right)\sqrt{\frac{2}{N}}\langle T\mathbf{z}, \mathbf{e}_c(m)\rangle + \cos\left(\frac{2\pi m}{N}\right)\left(-\sqrt{\frac{2}{N}}\langle T\mathbf{z}, \mathbf{e}_s(m)\rangle\right).$$

Operating, the following equalities are obtained:

$$\sqrt{\frac{2}{N}}\langle T\mathbf{z}, \mathbf{e}_c(m)\rangle = \cos\left(\frac{2\pi m}{N}\right) q_m \cos\varphi_m - \sin\left(\frac{2\pi m}{N}\right) q_m \sin\varphi_m,$$
$$\sqrt{\frac{2}{N}}\langle T\mathbf{z}, \mathbf{e}_s(m)\rangle = \cos\left(\frac{2\pi m}{N}\right) q_m \sin\varphi_m + \sin\left(\frac{2\pi m}{N}\right) q_m \cos\varphi_m,$$

i.e.

$$\sqrt{\frac{2}{N}}\langle T\mathbf{z}, \mathbf{e}_c(m)\rangle = q_m \cos\left(\varphi_m + \frac{2\pi m}{N}\right), \tag{28}$$
$$-\sqrt{\frac{2}{N}}\langle T\mathbf{z}, \mathbf{e}_s(m)\rangle = q_m \sin\left(\varphi_m + \frac{2\pi m}{N}\right), \tag{29}$$

which implies

$$Tq_m = q_m, \quad T\varphi_m = \varphi_m + \frac{2\pi m}{N}.$$

The proof of point 1 is ended for the case when N is even, by noting that $T\mathbf{e}_c(N/2) = -\mathbf{e}_c(N/2)$, which readily implies that $Tq_{N/2} = -q_{N/2}$.
Proof of point 2. Combining the definition of the puckering coordinates with relation (6), the following equalities are obtained

$$\begin{aligned} Dq_m \cos D\varphi_m &= \sqrt{\frac{2}{N}}\langle D\mathbf{z}, \mathbf{e}_c(m)\rangle = -\sqrt{\frac{2}{N}}\langle D\mathbf{z}, D\mathbf{e}_c(m)\rangle, \\ Dq_m \sin D\varphi_m &= -\sqrt{\frac{2}{N}}\langle D\mathbf{z}, \mathbf{e}_s(m)\rangle = -\sqrt{\frac{2}{N}}\langle D\mathbf{z}, D\mathbf{e}_s(m)\rangle. \end{aligned}$$

Since $\langle D\mathbf{z}, D\mathbf{e}_c(m)\rangle = \langle \mathbf{z}, \mathbf{e}_c(m)\rangle$ and $\langle D\mathbf{z}, D\mathbf{e}_s(m)\rangle = \langle \mathbf{z}, \mathbf{e}_s(m)\rangle$, it is deduced that

$$\begin{aligned} Dq_m \cos D\varphi_m &= -q_m \cos \varphi_m, \\ Dq_m \sin D\varphi_m &= q_m \sin \varphi_m, \end{aligned}$$

which implies

$$Dq_m = q_m, \quad D\varphi_m = \pi - \varphi_m.$$

Moreover, if N is even, it follows from $D\mathbf{e}_c(N/2) = -\mathbf{e}_c(N/2)$ that $q_{N/2} = -q_{N/2}$.
Proof of point 3. Following the same reasoning as for point 2, it holds that

$$\begin{aligned} Mq_m \cos M\varphi_m &= -q_m \cos \varphi_m, \\ Mq_m \sin M\varphi_m &= -q_m \sin \varphi_m, \end{aligned}$$

which implies

$$Dq_m = q_m, \quad D\varphi_m = \pi + \varphi_m.$$

Moreover, if N is even, it follows from $M\mathbf{e}_c(N/2) = -\mathbf{e}_c(N/2)$ that $q_{N/2} = -q_{N/2}$.

A.2.2 Proof of Propositions 3–7

The proofs of Propositions 3– 7 follow the same lines.

They consist in proving on one hand that any composition $\prod_{s=1}^{r} M^{k_s} D^{j_s} T^{i_s}$, with $(i_s, j_s) \in \mathscr{S}$ and $k_s \in \{0, 1\}$, for $s = 1, \ldots, r$ can be written in the form $M^K D^J T^I$ for some $K \in \{0, 1\}$ and $(I, J) \in \mathscr{S}$, and on the other hand, in obtaining all equivalent pairs or triplets by making (I, J, K) vary in $\mathscr{S} \times \{0, 1\}$ and collecting the results in a table like Tables 2–4.

To prove the first statement, it is clearly enough to do it for $r = 2$, i.e, it is to be checked that, for any (i_1, j_1, k_1) in $\mathscr{S} \times \{0, 1\}$ and (i_2, j_2, k_2) in $\mathscr{S} \times \{0, 1\}$, there exist (I, J, K) in $\mathscr{S} \times \{0, 1\}$ such that

$$M^{k_1} D^{j_1} T^{i_1} M^{k_2} D^{j_2} T^{i_2} = M^K D^J T^I.$$

This is the case since

$$M^{k_1} D^{j_1} T^{i_1} M^{k_2} D^{j_2} T^{i_2} = M^{k_1+k_2} D^{j_1+j_1} T^{i_2+(-1)^{j_2} i_1}.$$

References

1. F. Allen, The Cambridge structural database: a quarter of a million crystal structures and rising. Acta Cryst. B **58**, 380–388 (2002)
2. F. Allen, J. Howard, N. Pitchford, Symmetry-modified conformational mapping and classification of the medium rings from crystallographic data. I. Cycloheptane. Acta Cryst. B **49**, 910–928 (1993)
3. F. Allen, J. Howard, N. Pitchford, Symmetry-modified conformational mapping and classification of the medium rings from crystallographic data. IV. Cyclooctane and related eight-membered rings. Acta Cryst. B **52**, 882–891 (1996)
4. D. Cremer, J. Pople, A general definition of ring puckering coordinates. J. Am. Chem. Soc. **97**(6), 1354–1358 (1975)
5. D. Evans, J. Boeyens, Mapping the conformation of eight-membered rings. Acta Cryst. B **44**, 663–671 (1988)
6. M. Kessler, M. Bueso, J. Pérez, Model-based conformational clustering of ring molecules. J. Chemom. **21**, 53–64 (2007)
7. M. Kessler, J. Pérez , M.C. Bueso, L. García, E. Pérez, J.L. Serrano, R. Carrascosa, Probabilistic model-based methodology for the conformational study of cyclic systems: application to copper complexes double-bridged by phosphate and related ligands. Acta Crystallogr Sect B **63**(6), 869–878 (2007). doi:10.1107/S0108768107050859. URL http://dx.doi.org/10.1107/S0108768107050859
8. J. Pérez, K. Nolsøe, M. Kessler, L. García, E. Pérez, J.L. Serrano, Bayesian methods for the conformational classification of eight-membered rings. Acta Cryst. B **61**, 585–594 (2005)